Journal of Engineering Mathematics, Vol. 12, No. 2, April 1978
Sijthoff & Noordhoff International Publishers–Alphen aan den Rijn
Printed in The Netherlands

Multiparameter families of difference approximations for the first initial boundary value problem for the heat equation in an arbitrary region ⋆

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(Received January 18, 1977)

SUMMARY

A symbolic technique is developed to automatically generate consistent multiparameter families of difference approximations to the heat equation with Dirichlet boundary conditions in arbitrary regions. A stencil of non-uniform step size, conformable along the spatial axes via spatial displacement parameters, is devised to handle the problem of irregular boundaries. Using this stencil as a basic building block, multiparameter families of difference schemes applicable without modification both in the interior and along the boundaries of arbitrary regions, are algorithmically generated. The technique is demonstrated in detail for the one- and two-dimensional heat operator. Necessary and sufficient conditions for the stability of these families are given in terms of their parameters. All existing six- and ten-point two-level schemes for the one- and two-dimensional cases are shown to form subclasses of these families.

1. Introduction

Many scientific problems are formulated as partial differential equations. Since very few of these can be solved analytically, various techniques have been devised for obtaining approximate solutions [1, 2, 3, 9, 11, 12]. Among the large number of numerical methods proposed for solving partial differential equations, the method of finite differences has particular importance because of its universal applicability to both linear and nonlinear problems. While the repetitive nature of the method makes it particularly well-suited for digital computations, time and space complexities are often quite large. The search for efficient finite difference methods has, therefore, been intensive over the past decades, resulting in the development of many schemes. However, until recently there has been no unified approach for generating and testing new difference schemes. Each formula has had to be considered individually with respect to properties such as accuracy, consistency, stability and convergence.

Recently, Khalil [7] proposed an algorithm for generating consistent families of difference approximations that depend on several parameters. This was illustrated by deriving and analyzing a two-parameter, eighteen-point, two-level family of high-order approximations

⋆ This work was supported in part by the UNIDEL Foundation at the University of Delaware and in part by grants to the Massachusetts Institute of Technology MATHLAB Project from the Advanced Research Projects Agency (ARPA) and the Department of Defense under Office of Naval Research Contract N00014-70-A-0362-001.

to the two-dimensional equation of heat flow in a polygonal region with Dirichlet boundary conditions. Later, Giese and Khalil [8] used inverse Vandermonde matrices to formalize the approach and derived a twelve-parameter family of eighteen-point, two-level approximations for the same equation. With this unifying approach, families can be analyzed in terms of their parameters to determine methods of increased efficiency.

A natural extension of their work is to generalize the approach to handle equations in arbitrary regions. Embedding these generalizations in a multiparameter family of difference methods will undoubtedly be profitable, since it enables us to study the effects of boundary contours on the choice of parameters. Moreover, the algorithm thus developed will form the nucleus of software for solving partial differential equations.

The desired generalization was explored using the Vandermonde matrix formalization. This led to a numeric approach in which each specific case had to be handled individually, resulting in an increase of complexity in the generation of families. Although the increase in complexity for a specific case is not significant, the number of cases to be considered grows rapidly with the number of spatial coordinates.

It therefore seemed advantageous to vary the approach in such a way as to minimize the complexity but still retain the technique of automatically generating consistent multiparameter families. A symbolic approach, based on a net of variable geometry, was selected for this purpose. In addition to generation parameters, explicit spatial displacement parameters are used to achieve non-uniformity of step size along the spatial axes. This device permits algorithmic generation of multiparameter families of difference schemes applicable without modification both in the interior and along the boundaries of general regions defined by sufficiently smooth boundaries, thus minimizing complexity. In addition, such an approach permits apriori stability analysis to yield valuable information regarding the choice of generation parameters. Finally, this approach provides a unifying principle for independently developed methods: all the well-known two-level approximations, as well as the five and twelve-parameter families of Giese and Khalil, are embedded in the approximations derived here.

In Section 2 we develop and analyze a multiparameter family of non-uniform difference methods for the one-dimensional equation of heat flow with Dirichlet boundary conditions. Section 3 deals with the extension of the method to higher dimensions with Dirichlet boundary conditions. As a specific case, a ten-point, two-level multiparameter family is studied in detail.

2. A non-uniform multiparameter family of difference analogues to $u_t = u_{xx}$

The simplest technique for solving a partial differential equation numerically is the finite difference method. Our aim is to automatically generate a non-uniform analogue to the heat operator which will serve as a nucleus in the development of computer software for the solution of this class of problems. To this end we seek the most general analogue. Consequently, our approach is to construct families of methods which depend on several parameters: spatial displacement parameters (SDP) and generation parameters (GP). Assignment of values to these parameters yields various methods which may be used as circumstances dictate.

To illustrate this idea we shall consider in this section the one-dimensional heat equation

$$u_t = u_{xx} \tag{2.1}$$

in an arbitrary region R of the (x, t) space with smooth boundary ∂R, subject to initial conditions

$$u(x, 0) = f(x) \text{ on the } x\text{-axis}$$

and Dirichlet boundary conditions $u(x, t) = g(x, t)$ on ∂R for prescribed functions f and g. We assume that a solution exists and that both the solution and the boundaries are sufficiently smooth and differentiable.

To solve this problem using finite differences, we cover the region $R \times \partial R$ by a lattice of discrete points and approximate $u(x, t)$ by the difference operator

$$\begin{aligned} L[u(x,t)] = {} & Au(x - \alpha_n h, t - \tfrac{1}{2}k) + Bu(x, t - \tfrac{1}{2}k) \\ & + Cu(x + \beta_n h, t - \tfrac{1}{2}k) + Du(x - \alpha_{n+1} h, t + \tfrac{1}{2}k) \\ & + Eu(x, t + \tfrac{1}{2}k) + Fu(x + \beta_{n+1} h, t + \tfrac{1}{2}k) \end{aligned} \tag{2.2}$$

where

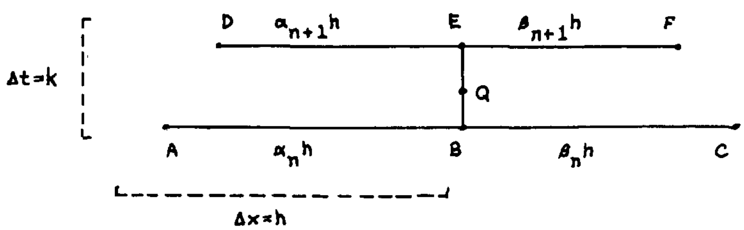
$$h = \Delta x > 0, \ k = \Delta t > 0, \text{ and } 0 < \alpha, \beta \le 1. \tag{2.3}$$

Here α_n, β_n, α_{n+1}, β_{n+1} denote the left and right SDP at the lower and upper time levels respectively.

Unlike familiar difference operators, (2.2) generates a non-uniform lattice.

2.1. *The fundamental stencil*

The set of coefficients $\mathscr{C} = \{A, B, C, D, E, F\}$ and center point $Q = (x, t)$ define the stencil $S(Q)$ associated with the difference operator (2.2). We say that this stencil is *conformable*, in that nodes $A, C, D, F \in \mathscr{C}$ can be positioned to lie directly on ∂R. The vehicle permitting this conformability is the set $\mathscr{A}$ of SDP, $\{\alpha_n, \beta_n, \alpha_{n+1}, \beta_{n+1}\}$. The stencil is represented by the diagram below:



To compute the coefficients $\mathscr{C}$ of the difference operator (2.2), we expand $L[u(x, t)]$ about the central point (x, t) as a Taylor's series in powers of h and k to obtain

$$\begin{aligned} L[u(x,t)] = {} & \sum_{p,q=0}^{\infty} [(-\alpha_n)^p(-\tfrac{1}{2})^q A + (-\tfrac{1}{2})^q B \\ & + (\beta_n)^p(-\tfrac{1}{2})^q C + (-\alpha_{n+1})^p(\tfrac{1}{2})^q D + (\tfrac{1}{2})^q E \\ & + (\beta_{n+1})^p(\tfrac{1}{2})^q F] \cdot \frac{h^p k^q}{p!\,q!} \cdot \frac{\partial^{p+q} u}{\partial x^p \partial t^q}. \end{aligned} \tag{2.4}$$

We select (p, q) pairs to include the explicit terms of the heat operator and force the u and u_x terms in (2.4) to vanish. The four conditions thus imposed for the (p, q) pairs (0, 0), (1, 0), (2, 0) and (0, 1) will restrict the choices of the coefficient set $\mathscr{C}$ to a two-GP, four-SDP family. To uniquely define $\mathscr{C}$, we adjoin two additional (p, q) pairs, (1, 1) and (2, 1), which contain the two desired generation parameters, ξ and η. This yields the system of linear symbolic equations:

$$Qx = f \tag{2.5}$$

where

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -\alpha_n & 0 & \beta_n & -\alpha_{n+1} & 0 & \beta_{n+1} \\ \alpha_n^2 & 0 & \beta_n^2 & \alpha_{n+1}^2 & 0 & \beta_{n+1}^2 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}\alpha_n & 0 & -\frac{1}{2}\beta_n & -\frac{1}{2}\alpha_{n+1} & 0 & \frac{1}{2}\beta_{n+1} \\ -\frac{1}{2}\alpha_n^2 & 0 & -\frac{1}{2}\beta_n^2 & \frac{1}{2}\alpha_{n+1}^2 & 0 & \frac{1}{2}\beta_{n+1}^2 \end{bmatrix}$$

$x = (A\ B\ C\ D\ E\ F)^T$, $f = (0\,0 - 2h^{-2}\,k^{-1}\,\xi\,\eta)^T$.

Thus, from (2.1), (2.2), and (2.5)

$$L[u(x, t)] = u_t - u_{xx} + \xi hku_{tx} + \eta h^2 ku_{txx} + O(h^3) + O(k^2) \tag{2.6}$$

We are, of course, ultimately concerned with solving the difference equations generated by $L[u(x, t)]$ by computer. We could store system (2.5) and numerically determine the values of the set of coefficients $\mathscr{C}$ for a given (ξ, η) pair and each new value assignment to the SDP set $\mathscr{A}$. This would involve the numerical solution of a 6×6 system of equations twice/sweep in the one-dimensional case; in the higher dimensional cases, the number of solutions required per sweep would increase greatly. We could design much more efficient software if we could solve for the coefficients explicitly. Moreover, an explicit solution would allow us to analytically investigate the effect of our GP on the stability, local accuracy, etc. of our difference methods. The task of solving the system of equations (2.5) by hand would be too tedious and error-prone. Thus we have a good candidate for solution by an algebraic symbol manipulating system. The problem was solved symbolically by the MACSYMA [7] system, yielding the following solution:

$$\begin{bmatrix} A \\ B \\ C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} -[\eta - \beta_n\xi + h^{-2}]/[\alpha_n\beta_n + \alpha_n^2] \\ [\eta + \xi(\alpha_n - \beta_n) - k^{-1}\alpha_n\beta_n + h^{-2}]/\alpha_n\beta_n \\ -[\eta + \alpha_n\xi + h^{-2}]/[\alpha_n\beta_n + \beta_n^2] \\ [\eta - \beta_{n+1}\xi - h^{-2}]/[\alpha_{n+1}\beta_{n+1} + \alpha_{n+1}^2] \\ -[\eta + \xi(\alpha_{n+1} - \beta_{n+1}) - k^{-1}\alpha_{n+1}\beta_{n+1} - h^{-2}]/\alpha_{n+1}\beta_{n+1} \\ [\eta + \alpha_{n+1}\xi - h^{-2}]/[\alpha_{n+1}\beta_{n+1} + \beta_{n+1}^2] \end{bmatrix} \tag{2.7}$$

At present, the inversion of symbolic matrices of even this order of magnitude can be troublesome. The manipulation can be simplified by taking advantage of the fact that the matrix can be partitioned in the form

$$Q = \left[\begin{array}{c|c} V_1 & V_2 \\ \hline -\frac{1}{2}V_1 & \frac{1}{2}V_2 \end{array}\right]$$

where V_1 and V_2 are Vandermonde matrices of order 3, and whose inverses are easily computed [5]. Then

$$Q^{-1} = \left[\begin{array}{c|c} \frac{1}{2}V_1^{-1} & -V_1^{-1} \\ \hline \frac{1}{2}V_2^{-1} & V_2^{-1} \end{array}\right].$$

2.2. *Matrix formulation*

Let $u_i^n = u(x_i, t_n)$, where $0 \le i \le N + 1$, $n \ge 0$. We approximate u_i^n by solutions U_i^n of the linear system

$$M_{n+1}U^{n+1} = M_n U^n + Z^n \tag{2.8}$$

where $U^s = (U_1^s, U_2^s, \ldots, U_N^s)^T$ and $s = n, n + 1$; M_{n+1} and M_n are square tridiagonal matrices of order N whose elements are functions of the SDP and GP, and are given by

$$M_{n+1} = \begin{bmatrix} E & F & & & \mathbf{0} \\ D & E & F & & \\ & \ddots & \ddots & \ddots & \\ & & D & E & F \\ \mathbf{0} & & & D & E \end{bmatrix}, \quad M_n = -\begin{bmatrix} B & C & & & \mathbf{0} \\ A & B & C & & \\ & \ddots & \ddots & \ddots & \\ & & A & B & C \\ \mathbf{0} & & & A & B \end{bmatrix}, \tag{2.9}$$

Z^n is a vector of N components involving the boundary conditions.

We can express (2.9) more conveniently as the weighted sum of simpler matrices I, W_n, W_{n+1}, Y_n, and Y_{n+1}:

$$\begin{aligned} M_{n+1} &= k^{-1}I + \tfrac{1}{2}(h^{-2} - \eta)W_{n+1} - \tfrac{1}{2}\xi Y_{n+1}, \\ M_n &= k^{-1}I - \tfrac{1}{2}(h^{-2} + \eta)W_n - \tfrac{1}{2}\xi Y_n. \end{aligned} \tag{2.10}$$

Let $s = n, n + 1$, and $1 \le i, j \le N$. Then the elements in the i^{th} row and j^{th} column of W_s and Y_s are:

$$W_s = \begin{cases} 2/\alpha_s\beta_s & \text{if } i = j \\ -2/[\alpha_s(\alpha_s + \beta_s)] & \text{if } i - j = 1 \\ -2/[\beta_s(\alpha_s + \beta_s)] & \text{if } j - i = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.11}$$

$$Y_s = \begin{cases} [2(\alpha_s - \beta_s)]/\alpha_s\beta_s & \text{if } i = j \\ 2\beta_s/[\alpha_s(\alpha_s + \beta_s)] & \text{if } i - j = 1 \\ -2\alpha_s/[\beta_s(\alpha_s + \beta_s)] & \text{if } j - i = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.12}$$

I denotes the $N \times N$ unit matrix.

As usual, in the interior of the region R, we will work with a uniform net. Moreover, for $N \geq 2$, it is reasonable to assign the value one to β_n and β_{n+1} in row 1 (at the left-hand boundary) and to α_n and α_{n+1} in row N (the right-hand boundary). Then (2.11) and (2.12) have the forms

$$W_s = \begin{bmatrix} 2/\alpha_s & -2/(1+\alpha_s) & & & \mathbf{0} \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ \mathbf{0} & & & -2/(1+\beta_s) & 2/\beta_s \end{bmatrix} \tag{2.13}$$

$$Y_s = \begin{bmatrix} 2(\alpha_s - 1)/\alpha_s & -2\alpha_s/(1+\alpha_s) & & & \mathbf{0} \\ 1 & 0 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & -1 \\ \mathbf{0} & & & 2\beta_s/(\beta_s + 1) & 2(1-\beta_s)/\beta_s \end{bmatrix} \tag{2.14}$$

For future use, we shall need some information about the eigenvalues of W_s and Y_s. First, we note that $\tilde{W}_s = H_s^{-1} W_s H_s$ is symmetric, where

$$H_s = \begin{cases} \sqrt{2/(1+\alpha_s)} & \text{if } i = j = 1 \\ \sqrt{2/(1+\beta_s)} & \text{if } i = j = N \\ \delta_{ij} & \text{otherwise} \end{cases} \tag{2.15}$$

$s = n, n+1$ and δ_{ij} denotes the Kronecker delta of rank 2.

Hence, the eigenvalues of $\tilde{W}_s$, and therefore those of W_s, are real.

The matrix W_s is irreducible, with diagonal dominance assured for rows 2 through $N-1$. Strict inequality occurs for rows 1 and N when

$$\left|\frac{2}{\alpha_s}\right| > \left|\frac{-2}{1+\alpha_s}\right| \quad \text{and} \quad \left|\frac{2}{\beta_s}\right| > \left|\frac{-2}{1+\beta_s}\right|. \tag{2,16}$$

respectively. By (2.3), these conditions are always met. Thus, the eigenvalues μ_s of W_s are always positive. Using Gerschgorin's Theorem [13] to derive an upper bound, we find

$$0 < \mu_s \leq \mu_s^* \tag{2.17}$$

where

$$\mu_s^* = \max\left[4, \frac{2(2\alpha_s + 1)}{\alpha_s(\alpha_s + 1)}, \frac{2(2\beta_s + 1)}{\beta_s(\beta_s + 1)}\right]. \tag{2.18}$$

The matrix Y_s is skew-symmetric, with eigenvalues occurring as imaginary pairs and zero.

2.3. *Stability*

We will examine the stability of the difference operator using the matrix method for analyzing stability which automatically includes the effects of the boundaries. If M_{n+1} is nonsingular, (2.8) implies

$$U^{n+1} = M_{n+1}^{-1} M_n U^n + M_{n+1}^{-1} Z^n. \tag{2.19}$$

A necessary and sufficient condition for stability is that the spectral radius ρ of the amplification matrix $M_{n+1}^{-1} M_n$ be less than or equal to unity [1, 4, 9, 11]. The vector $M_{n+1}^{-1} Z_n$, does not effect stability.

Because of the asymmetry of the matrix Y_s, it is quite difficult to obtain a good estimate for $\rho\,(M_{n+1}^{-1} M_n)$. We can, however, obtain a reasonable estimate by setting $\xi = 0$. This has the effect of setting $A = C$ and $D = F$ in the interior of R. Let us further simplify the analysis by letting $\sigma = \min[\alpha_s, \beta_s]$, and let W, Y, H, and μ^* denote W_s, Y_s, H_s, and μ_s^*, respectively, with α_s and β_s replaced by σ.

Then

$$M_{n+1}^{-1} M_n = [k^{-1} I + \tfrac{1}{2}(h^{-2} - \eta)W]^{-1}[k^{-1} I - \tfrac{1}{2}(h^{-2} + \eta)W]. \tag{2.20}$$

By straightforward substitution we find

$$\Lambda = \frac{k^{-1} - \frac{1}{2}(h^{-2} + \eta)\mu}{k^{-1} + \frac{1}{2}(h^{-2} - \eta)\mu} \tag{2.21}$$

where $\Lambda \in \mathscr{M}$, the set of eigenvalues of $M_{n+1}^{-1} M_n$ and $\mu \in \mathscr{W}$, the set of eigenvalues of W.

Moreover, using (2.16) we see that

$$\begin{aligned}
\widetilde{(M_{n+1}^{-1} M_n)} &= H^{-1}(M_{n+1}^{-1} M_n)H \\
&= H^{-1}[k^{-1} I + \tfrac{1}{2}(h^{-2} - \eta)W]^{-1}[k^{-1} I - \tfrac{1}{2}(h^{-2} + \eta)W]H \\
&= \{H^{-1}[k^{-1} I + \tfrac{1}{2}(h^{-2} - \eta)W]^{-1}H\}\{H^{-1}[k^{-1} I - \tfrac{1}{2}(h^{-2} + \eta)W]H\} \\
&= \{H^{-1}[k^{-1} I + \tfrac{1}{2}(h^{-2} - \eta)W]H\}^{-1}\{H^{-1}[k^{-1} I - \tfrac{1}{2}(h^{-2} + \eta)W]H\} \\
&= [k^{-1} I + \tfrac{1}{2}(h^{-2} - \eta)\tilde{W}]^{-1}[k^{-1} I - \tfrac{1}{2}(h^{-2} + \eta)\tilde{W}].
\end{aligned} \tag{2.22}$$

Since matrices $[k^{-1} I + \frac{1}{2}(h^{-2} - \eta)\tilde{W}]^{-1}$ and $[k^{-1} I - \frac{1}{2}(h^{-2} + \eta)\tilde{W}]$ are symmetric and commute, $\widetilde{(M_{n+1}^{-1} M_n)}$ is symmetric. Thus $(M_{n+1}^{-1} M_n)$ is similar to a symmetric matrix so that

$$\rho(M_{n+1}^{-1} M_n) \equiv \rho(\widetilde{M_{n+1}^{-1} M_n}) \leq 1 \tag{2.23}$$

is a necessary and sufficient condition for stability. We insist then that $|\Lambda| \leq 1$ for all $\Lambda \in \mathscr{M}$, which leads to the following conditions:

$$\text{(i)}\quad \Lambda \le 1 \qquad \text{if } -h^{-2}\mu \le 0, \tag{2.24}$$

$$\text{(ii)}\quad \Lambda \ge -1 \qquad \text{if } \eta\mu \le 2k^{-1}. \tag{2.25}$$

Condition (i) is trivially satisfied since h and μ are always positive. From (2.17) we find that a necessary and sufficient condition for the unconditional stability of the family defined by (2.2), (2.7), and $\xi = 0$, is that η and k satisfy the following condition:

$$\eta \le \frac{2}{k\mu^*}, \tag{2.26}$$

where

$$\mu^* = \begin{cases} \dfrac{2(2\sigma+1)}{\sigma(\sigma+1)} & \text{for } 0 < \sigma < \sqrt{\tfrac{1}{2}} \\ 4 & \text{for } \sqrt{\tfrac{1}{2}} \le \sigma \le 1 \end{cases}$$

In the limit, this condition takes the form

$$\lim_{\sigma\to 0} \eta \le 0, \tag{2.27}$$

which assures unconditional stability independent of the values of the SDP.

TABLE 1
Familiar two-level difference operators

Scheme	ξ	η	A	B	C	D	E	F	Stability conditions
Classical explicit	0	h^{-2}	$-h^{-2}$	$2h^{-2}-k^{-1}$	$-h^{-2}$	0	k^{-1}	0	$kh^{-2} \le \frac{1}{2}$
Classical implicit	0	$-h^{-2}$	0	$-k^{-1}$	0	$-h^{-2}$	$2h^{-2}+k^{-1}$	$-h^{-2}$	none
Crank–Nicolson	0	0	$-\frac{1}{2}h^{-2}$	$-k^{-1}+h^{-2}$	$-\frac{1}{2}h^{-2}$	$-\frac{1}{2}h^{-2}$	$k^{-1}+h^{-2}$	$-\frac{1}{2}h^{-2}$	none
One-parameter	0	$h^{-2}(1-2\theta)$	$(\theta-1)h^{-2}$	$-k^{-1}$ $+2(1-\theta)h^{-2}$	$(\theta-1)h^{-2}$	$-\theta h^{-2}$	$k^{-1}+2\theta h^{-2}$	$-\theta h^{-2}$	$kh^{-2}(1-2\theta)$ $\le \frac{1}{2}$
Asymmetric	h^{-2} $-h^{-2}$	0 0	0 $-h^{-2}$	$-k^{-1}+h^{-2}$ $-k^{-1}+h^{-2}$	$-h^{-2}$ 0	$-h^{-2}$ 0	$k^{-1}+h^{-2}$ $k^{-1}+h^{-2}$	0 $-h^{-2}$	none, if used on alternate time steps

If we restrict the region to the usual rectangular shape, we can set all the SDP to unity to achieve a uniform net throughout R. The necessary and sufficient condition for stability for arbitrary η and k with $\xi = 0$, then, is given by

$$\eta \leq \tfrac{1}{2}k^{-1}. \tag{2.28}$$

Since $M_{n+1}^{-1}M_n$ is a continuous rational function of the GP, we can expect that its eigenvalues will also be continuous functions of the GP. This being the case, we can generalize our results to the extent of predicting that methods satisfying the stability criterion (2.26) will also be unconditionally stable for sufficiently small values of $|\xi|$. By Lax's equivalence theorem [6], immediately applicable for the single equation (2.1), stability of our difference methods is equivalent to the convergence of U_i^n to $u(x, t)$ in R.

2.4. *Generation of standard two-level difference operators*

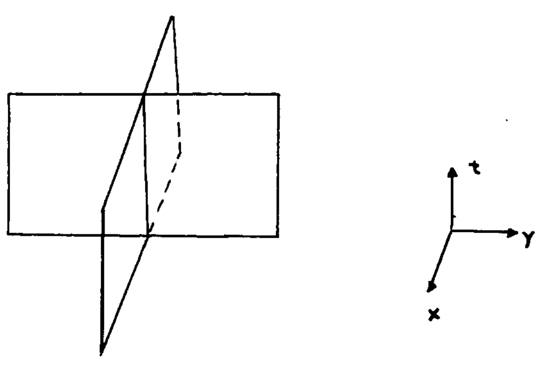
The familiar two-level approximations [11, 12] displayed in Table 1, can be obtained from our method by specializing the generation parameters ξ and η after setting all SDP to unity. The local accuracy and stability criteria of schemes in which $\xi = 0$ is immediately apparent from (2.6) and (2.28).

3. Equations of higher dimensions

In this section we consider a technique for using the conformable stencil developed for the one-dimensional heat equation to build lattices for higher order heat equations subject to Dirichlet boundary conditions in an arbitrary region. Since most of the additional complications arise immediately with the addition of one more space dimension, we content ourselves with considering only the two-dimensional case in detail. The further extension to three or more space dimensions appears to be straightforward.

3.1. *Extension of method to higher dimensions*

A natural extension of the approach developed for the one-dimensional case can be thought of in geometric terms as the intersection of two conformable stencils. The stencil for the one-dimensional heat equation consisted of six points in the x, t-plane; in two dimensions it is natural to combine a six-point stencil in a plane normal to the y-axis with another six-point stencil in a plane normal to the x-axis, insisting that the centered points coincide. This results in a ten-point stencil, illustrated below:



The difference operator associated with this stencil can be formed as the algebraic sum of two judiciously modified one-dimensional operators. Let $L_x[u]$ and $L_y[u]$ be the operators associated with the stencils in the planes normal to the x- and y-axes, respectively. Instead of approximating the standard heat equation, we approximate $\frac{1}{2}u_t - u_{xx}$ by $L_x[u]$ and $\frac{1}{2}u_t - u_{yy}$ by $L_y[u]$. We can easily specify these operators by making a slight change in the right-hand vector of the system of equations defining the stencil coefficients (2.5). The component k^{-1} in the vector f is replaced by $\frac{1}{2}k^{-1}$. The effect of this replacement is a simple change of variable in the solution vector given by (2.7). The difference operator associated with the ten-point stencil, $L_{10}[u]$, can then be written as

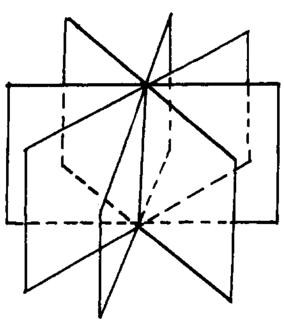
$$\begin{aligned} L_{10}[u] &= L_x[u] + L_y[u] \\ &= u_t - u_{xx} - u_{yy} + \xi_x hku_{tx} + \eta_x h^2ku_{txx} \\ &\quad + \xi_y hku_{ty} + \eta_y h^2 ku_{tyy} + O(h^3) + O(k^2) \\ &= \sum A_{a,b} u(x \pm ah, y \pm bh, t \pm \tfrac{1}{2}k) \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} a &= 0, \alpha_n^x, \alpha_{n+1}^x, \beta_n^x, \beta_{n+1}^x, \\ b &= 0, \alpha_n^y, \alpha_{n+1}^y, \beta_n^y, \beta_{n+1}^y, 0 < \alpha, \beta \leq 1. \end{aligned}$$

Thus, $L_{10}[u]$ defines a two-level, ten-point, four-GP, eight-SDP family of difference approximations to the two-dimensional heat equation whose coefficients, $A_{a,b}$, are explicitly known.

Extending this approach further, we combine two pairs of mutually perpendicular conformable stencils — one pair oriented coincident with the coordinate axes, the second pair rotated 45° about the t-axis — to produce an eighteen-point stencil:



Let $L_x^\phi[u]$ and $L_y^\phi[u]$ be the operators associated with the rotated stencil-pair. Since each pair of stencils approximates the two-dimensional heat-equation, we define our eighteen-point operator as

$$L_{18}[u] = \tfrac{1}{2}[L_x + L_x^\phi + L_y + L_y^\phi]. \tag{3.2}$$

The operator $L_{18}[u]$ defines a two-level, eighteen-point, eight-GP, sixteen-SDP family of difference approximations.

In general, the intersection of n conformable stencils in the (x, y, t) space leads to a variety of two-level, $(4n + 2)$-point, $2n$-GP, $4n$-SDP family of finite difference approximations to the two-dimensional heat operator. The coefficients of the difference operators can be obtained in explicit form, after some manipulation, from (2.7).

3.2. *Analysis of a ten-point multiparameter family for* $u_t = u_{xx} + u_{yy}$

Let $R = J \times I$ be an arbitrary region of the (x, y, t) space with smooth boundary R, where J is a smoothly bounded region in the xy plane and I is the interval $0 \leq t \leq T \leq \infty$. We consider the two-dimensional heat equation

$$u_t = u_{xx} + u_{yy} \tag{3.3}$$

subject to initial conditions

$$u(x, y, 0) = f(x, y), \qquad (x, y) \in J$$

and the boundary conditions

$$u(x, y, t) = g(x, y, t), \qquad (x, y, t) \in \partial R.$$

3.2.1. *Matrix formulation*

Let $u^n_{i,j} = u(x_i, y_j, t_n)$, where $i, j, n \geq 0$. We approximate $u^n_{i,j}$ by solutions $U^n_{i,j}$ of the linear system

$$M_{n+1} U^{n+1} = M_n U^n + Z^n. \tag{3.4}$$

U^n and U^{n+1} are N-dimension vectors; Z^n is an N-dimension vector involving boundary conditions; and M_{n+1} and M_n are square block tridiagonal matrices of order N, whose elements are the coefficients of the difference operator (3.1).

We anticipate from Section 2.2 that the matrices M_{n+1} and M_n can be expressed as the weighted sum of simpler matrices, I_N, $\bar{W}_n$, $\bar{W}_{n+1}$, $\bar{Y}_n$, and $\bar{Y}_{n+1}$, each block tridiagonal and of order N. Let $\bar{W}_s(I, J; i, j)$ and $\bar{Y}_s(I, J; i, j)$ denote the i^{th} row and j^{th} column of the (I, J) block of $\bar{W}_s$ and $\bar{Y}_s$, respectively, where $s = n, n + 1$. Then the entries of $\bar{W}_s$ and $\bar{Y}_s$ are shown in the table on p. 108.

Separating components contributed by the operators $L_x[u]$ and $L_y[u]$, we can express $\bar{W}_s$ and $\bar{Y}_s$ as

$$\begin{aligned} \bar{W}_s &= W^x_s + W^y_s, \\ \bar{Y}_s &= Y^x_s + Y^y_s. \end{aligned} \tag{3.5}$$

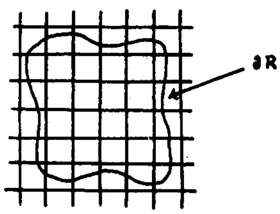
We can now express M_{n+1} and M_n as

$$\begin{aligned} M_{n+1} &= k^{-1} I_N + \tfrac{1}{2}(h^{-2} - \eta_x) W^x_{n+1} - \tfrac{1}{2}\xi_x Y^x_{n+1} + \tfrac{1}{2}(h^{-2} - \eta_y) W^x_{n+1} - \tfrac{1}{2}\xi_y Y^y_{n+1}, \\ M_n &= k^{-1} I_N - \tfrac{1}{2}(h^{-2} + \eta_x) W^x_n - \xi_x Y^x_n - \tfrac{1}{2}(h^{-2} + \eta_y) W^y_n - \tfrac{1}{2}\xi_y Y^y_n. \end{aligned} \tag{3.6}$$

		$\bar{W}_s$	$\bar{Y}_s$
$I = J$	$i = j$	$\frac{2}{\alpha_s^x \beta_s^x} + \frac{2}{\alpha_s^y \beta_s^y}$	$\frac{2(\alpha_s^x - \beta_s^x)}{\alpha_s^x \beta_s^x} + \frac{2(\alpha_s^y - \beta_s^y)}{\alpha_s^y \beta_s^y}$
	$i - j = 1$	$-\frac{2}{\alpha_s^y(\alpha_s^y + \beta_s^y)}$	$\frac{2\beta_s^y}{\alpha_s^y(\alpha_s^y + \beta_s^y)}$
	$j - i = 1$	$-\frac{2}{\beta_s^y(\alpha_s^y + \beta_s^y)}$	$-\frac{2\alpha_s^y}{\beta_s^y(\alpha_s^y + \beta_s^y)}$
	else	0	0
$I - J = 1$	$i = j$	$-\frac{2}{\alpha_s^x(\alpha_s^x + \beta_s^x)}$	$\frac{2\beta_s^x}{\alpha_s^x(\alpha_s^x + \beta_s^x)}$
	else	0	0
$J - I = 1$	$i = j$	$-\frac{2}{\beta_s^x(\alpha_s^x + \beta_s^x)}$	$-\frac{2\alpha_s^x}{\beta_s^x(\alpha_s^x + \beta_s^x)}$
	else	0	0

3.2.2. *Stability analysis*

For purposes of analysis, we shall assume that the step size is chosen in such a way that each block in M_{n+1} and M_n is of equal size, i.e., ∂R has the form illustrated below:



This assumption is not especially restrictive, since 1) unconditional stability implies total flexibility in selecting step size and 2) the boundedness of the derivatives of the solution requires a definite degree of smoothness of the boundaries.

Furthermore, let $\sigma = \min[\alpha, \beta]$, and let W_x, W_y, Y_x, and Y_y denote W_s^x, W_s^y, Y_s^x, and Y_s^y, respectively, with α_s^x, α_s^y, β_s^x, β_s^y replaced by σ.

We define two $\sqrt{N}$-order matrices W and Y of the form

$$W = \frac{1}{\sigma^2}\begin{bmatrix} 2 & -1 & & \mathbf{0} \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & -1 & 2 & -1 \\ \mathbf{0} & & -1 & 2 \end{bmatrix},$$

$$Y=\frac{1}{\sigma}\begin{bmatrix} 0 & -1 & & & \mathbf{0} \\ 1 & 0 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & -1 \\ \mathbf{0} & & & 1 & 0 \end{bmatrix}. \tag{3.7}$$

Then

$$W_x = W\otimes I,\quad W_y = I\otimes W,\quad Y_x = Y\otimes I,\quad Y_y = I\otimes Y, \tag{3.8}$$

where I denotes the unit matrix of order $\sqrt{N}$ and the symbol $\otimes$ signifies the Kronecker product.

A necessary and sufficient condition for stability is that

$$\rho(M_{n+1}^{-1}M_n)\leq 1. \tag{3.9}$$

As in the one-dimensional case, it is necessary to set $\xi_x = \xi_y = 0$ in order to obtain a reasonable estimate for ρ. Then

$$\begin{aligned} M_{n+1} &= k^{-1}I_N + \tfrac{1}{2}(h^{-2}-\eta_x)W_x + \tfrac{1}{2}(h^{-2}-\eta_y)W_y, \\ M_n &= k^{-1}I_N - \tfrac{1}{2}(h^{-2}+\eta_x)W_x - \tfrac{1}{2}(h^{-2}+\eta_y)W_y. \end{aligned} \tag{3.10}$$

Since

$$W_xW_y = (W\otimes I)(I\otimes W) = W\otimes W = W_yW_x \tag{3.11}$$

the matrices possess a common set of orthonormal eigenvectors v with the corresponding eigenvalues given by

$$W_xv = \mu v,\quad W_yv = \lambda v.$$

It is clear from (3.11) that

$$M_{n+1}M_n = M_nM_{n+1}, \tag{3.12}$$

which implies that

$$M_nM_{n+1}^{-1} = M_{n+1}^{-1}M_n. \tag{3.13}$$

Hence, using Frobenius' lemma [10], we can express the eigenvalues Λ of $M_{n+1}^{-1}M_n$ in the form

$$\Lambda = \frac{k^{-1}-\frac{1}{2}(h^{-2}+\eta_x)\mu-\frac{1}{2}(h^{-2}+\eta_y)\lambda}{k^{-1}+\frac{1}{2}(h^{-2}-\eta_x)\mu+\frac{1}{2}(h^{-2}-\eta_y)\lambda}. \tag{3.14}$$

Both M_{n+1}^{-1} and M_n are symmetric, since W is symmetric. From (3.13), we see that $M_{n+1}^{-1}M_n$ is also symmetric, so that the condition

$$|\Lambda| \leq 1 \text{ for all } \Lambda, \mu, \text{ and } \lambda \tag{3.15}$$

is a necessary and sufficient condition for the stability of (3.4). This leads to the following conditions:

$$\text{(i)} \quad \Lambda \leq 1 \quad \text{if } -h^{-2}(\mu + \lambda) \leq 0 \tag{3.16}$$

$$\text{(ii)} \quad \Lambda \geq -1 \text{ if } \mu\eta_x + \lambda\eta_y \leq 2k^{-1}. \tag{3.17}$$

Using Gerschgorin's Theorem, we find

$$0 < \mu, \lambda \leq 4/\sigma 2. \tag{3.18}$$

Since h, μ, and λ are always positive, condition (i) is trivially satisfied. Stability conditions for (ii) are

$$\begin{aligned}
\eta_x + \eta_y &< .5\sigma^2 k^{-1} \quad \text{for} \quad \eta_x, \eta_y > 0, \\
\eta_x &< .5\sigma^2 k^{-1} \quad \text{for} \quad \eta_x > 0 \wedge \eta_y \leq 0, \\
&< .5\sigma^2 k^{-1} \quad \text{for} \quad \eta_x \leq 0 \wedge \eta_y > 0,
\end{aligned} \tag{3.19}$$

unconditional stability for $\eta_x, \eta_y \leq 0$.

In the limit as $\sigma \to 0$, unconditional stability is assured only when

$$\eta_x, \eta_y \leq 0. \tag{3.20}$$

Again, by the argument of continuity, we can expect condition (3.19) to be necessary and sufficient for the unconditional stability of methods specified by sufficiently small values of $|\xi_x|$ and $|\xi_y|$.

3.2.3. *Special cases*

3.2.3.1. *Generation of standard two-level difference operators*

The familiar two-level approximations, displayed in Table 2, are special cases of the subclass of the family (3.6) with $\sigma = 1$. The local accuracy and stability criteria of each scheme is immediately apparent from (3.1) and (3.19).

3.2.3.2. *Split-formula schemes*

We now consider a variant of (3.10) to derive a multiparameter family of ADI methods. Let

$$\begin{aligned}
M_{n+1} &= [k^{-1}I_N + \tfrac{1}{2}(h^{-2} - \eta_x)W_x][k^{-1}I_N + \tfrac{1}{2}(h^{-2} - \eta_y)W_y] \\
M_n &= [k^{-1}I_N - \tfrac{1}{2}(h^{-2} + \eta_x)W_x][k^{-1}I_N - \tfrac{1}{2}(h^{-2} + \eta_y)W_y].
\end{aligned} \tag{3.21}$$

The split formulas

$$\begin{aligned}
[k^{-1}I_N + \tfrac{1}{2}(h^{-2} - \eta_x)W_x]U^{n+1*} &= [k^{-1}I_N - \tfrac{1}{2}(h^{-2} + \eta_y)W_y]U^n \\
[k^{-1}I_N + \tfrac{1}{2}(h^{-2} - \eta_y)W_y]U^{n+1} &= [k^{-1}I_N - \tfrac{1}{2}(h^{-2} + \eta_x)W_x]U^{n+1*}
\end{aligned} \tag{3.22}$$

are obtained from (3.21).

TABLE 1
Familiar cases

Scheme	ξ_x	ξ_y	η_x	η_y	M_{n+1}	M_n	Stability
Explicit	0	0	h^{-2}	h^{-2}	$k^{-1}I_N$	$k^{-1}I_N - h^{-2}(W_x + W_y)$	$kh^{-2} \leq \frac{1}{4}$
Crank–Nicholson	0	0	0	0	$k^{-1}I_N + \frac{1}{2}h^{-2}(W_x + W_y)$	$k^{-1}I_N - \frac{1}{2}h^{-2}(W_x + W_y)$	none

Since the matrices W_x and W_y are symmetric and commute, (3.11), it can be shown that each of the factors comprising M_{n+1}^{-1} and M_n are symmetric and possess a common set of orthogonal eigenvectors. Hence, we can express the eigenvalues Λ of $M_{n+1}^{-1}M_n$, from (3.21), in the form

$$\Lambda = \frac{[k^{-1} - \frac{1}{2}(h^{-2} + \eta_x)\mu][k^{-1} - \frac{1}{2}(h^{-2} + \eta_y)\lambda]}{[k^{-1} + \frac{1}{2}(h^{-2} - \eta_x)\mu][k^{-1} + \frac{1}{2}(h^{-2} - \eta_y)\lambda]} \tag{3.22}$$

where μ and λ are the eigenvalues of W_x and W_y, respectively. From (3.18), $0 < \mu, \lambda \leq 4/\sigma^2$. A necessary and sufficient condition for the stability of this family of ADI methods is that

$$|\Lambda| \leq 1 \text{ for all } \Lambda, \mu, \lambda. \tag{3.23}$$

This leads to the following conditions:

$$\text{(i)}\quad \Lambda \leq 1 \text{ if } \eta_x + \eta_y \leq \frac{2(\mu + \lambda)}{\mu\lambda k}, \tag{3.24}$$

$$\text{(ii)}\quad \Lambda \geq -1 \text{ if } \mu\eta_x + \lambda\eta_y - \tfrac{1}{2}k\mu\eta_x\eta_y \leq 2k^{-1} + \tfrac{1}{2}\mu\lambda k h^{-4}. \tag{3.25}$$

Note that if we choose $\eta_x \leq 0$ and $\eta_y \leq 0$, the method will be unconditionally stable, independent of boundary irregularities.

Special cases:
Peaceman–Rachford Method: $\sigma = 1$, $\eta_x = \eta_y = 0$,
Mitchell–Fairweather Method: $\sigma = 1$, $\eta_x = \eta_y = -\frac{1}{6}$.

From (3.24) and (3.25) it is clear that both these methods are unconditionally stable.

3.3. *An eighteen-point multiparameter family for* $u_t = u_{xx} + u_{yy}$

Extending the techniques demonstrated for the ten-point operator to the eighteen-point operator (3.2), we define

$$\bar{W} = 3W_x + W_y + W_x^\phi + W_y^\phi,\ \bar{Y} = Y_x + Y_y + Y_x^\phi + Y_y^\phi \tag{3.26}$$

where

$$W_x = W \otimes I,\ W_y = I \otimes W,\ W_x^\phi = E \otimes W,\ W_y^\phi = E^* \otimes W,$$

$$Y_x = Y \otimes I,\ Y_y = I \otimes Y,\ Y_x^\phi = E \otimes Y,\ Y_y^\phi = E^* \otimes Y.$$

The matrices W, Y, and I are defined by (3.7). E is the forward shift operator defined by the $\sqrt{N}$-order matrix

$$E = \begin{bmatrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \mathbf{0} & & & 0 \end{bmatrix}.$$

E^* is the backward shift operator defined by the $\sqrt{N}$-order matrix

$$E^* = \begin{bmatrix} 0 & & & \mathbf{0} \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ \mathbf{0} & & 1 & 0 \end{bmatrix}.$$

The difference scheme is then defined by the matrices

$$\begin{aligned} M_{n+1} = {} & k^{-1}I_N + \tfrac{3}{2}(h^{-2} - \eta_x)W_x - \tfrac{1}{2}\xi_x Y_x + \tfrac{1}{2}(h^{-2} - \eta_y)W_y \\ & - \tfrac{1}{2}\xi_y Y_y + \tfrac{1}{2}(h^{-2} - \eta_x^\phi)W_x^\phi - \tfrac{1}{2}\xi_x^\phi Y_x^\phi + \tfrac{1}{2}(h^{-2} - \eta_y^\phi)W_y^\phi - \tfrac{1}{2}\xi_y^\phi Y_y^\phi, \\ M_n = {} & k^{-1}I_N - \tfrac{3}{2}(h^{-2} + \eta_x)W_x - \tfrac{1}{2}\xi_x Y_x - \tfrac{1}{2}(h^{-2} + \eta_y)W_y \\ & - \tfrac{1}{2}\xi_y Y_y - \tfrac{1}{2}(h^{-2} + \eta_x^\phi)W_x^\phi - \tfrac{1}{2}\xi_x^\phi Y_x^\phi - \tfrac{1}{2}(h^{-2} + \eta_y^\phi)W_y^\phi - \tfrac{1}{2}\xi_y^\phi Y_y^\phi. \end{aligned}$$

3.4. *Introduction of extra parameters*

The following technique can be used to increase the number of GP associated with operators for the two-dimensional heat equation. Let a two-level, ten-point conformable stencil $\mathscr{L}[u(x, y, t)]$ be the fundamental operator, where

$$\mathscr{L}[u] = L_x[u] + L_y[u] + \theta_x u_x + \theta_y u_y + \theta_{xy} u_{xy}, \tag{3.28}$$

where $L_x[u]$ and $L_y[u]$ are the operators introduced in Section 3.1 and θ_x, θ_y, and θ_{xy} are additional GP. If we choose to approximate the heat equation with a ten-point operator of $O(h^3)$, as in (3.1), we require that

$$\theta_x u_x = \theta_y u_y = \theta_{xy} u_{xy} = 0$$

and no new GP have been added. However, if we choose to combine at least two of these new fundamental operators to construct an eighteen-point operator, we have

$$\begin{aligned} \mathscr{L}_{18}[u] &= \tfrac{1}{2}\{\mathscr{L}[u] + \mathscr{L}^\phi[u]\} \\ &= \tfrac{1}{2}\{L_x[u] + L_y[u] + L_x^\phi[u] + L_y^\phi[u] + (\theta_x + \theta_x^\phi)u_x \\ &\quad + (\theta_y + \theta_y^\phi)u_y + (\theta_{xy} + \theta_{xy}^\phi)u_{xy}\}, \end{aligned} \tag{3.29}$$

in which six additional GP have been introduced. Imposing the three constraints necessary to assure that our operator is $O(h^3)$ leaves us with an operator with three more GP than we had for $L_{18}[u]$. This new operator thus has a total of eleven GP, and is equivalent to the eleven-parameter operator discussed by Giese and Khalil [8].

In general, combining n fundamental operators $\mathscr{L}[u]$ results in a two-level, $(8n+2)$-point, $8n$-SDP, $(7n-3)$-GP family of difference operators. Moreover, it is obvious that judicious selection of fundamental operators can lead to multiparameter families with varying characteristics, such as three-level families.

4. Conclusions

The use of a conformable stencil to algorithmically generate families of difference approximations to the first boundary value problem for the heat equation has led to some useful results:

1. a procedure for constructing multiparameter families of difference schemes which contain boundary information for regions of arbitrary shape;
2. a simple method for determining the conditions for stability of each family symbolically, in terms of generative parameters and spatial displacement parameters;
3. a unified account of existing two-level difference approximations to the one- and two-dimensional heat operator.

We have demonstrated these techniques in detail for:

1. a two-level, six-point, two-GP, four-SDP family of approximations to the one-dimensional heat operator with Dirichlet boundary conditions in an arbitrary region;
2. a two-level, ten-point, four-GP, eight-SDP family of approximations to the two-dimensional heat operator with Dirichlet boundaries in an arbitrary region;
3. a two-level, ten-point, four-GP, eight-SDP family of ADI methods for the two-dimensional heat operator with Dirichlet boundaries in an arbitrary region.

Necessary and sufficient conditions for the stability of each of these families was determined symbolically in terms of their parameters; it was shown that proper choices of the generative parameters assured stability independent of boundary conditions. The well-known corresponding difference schemes were shown to be a subclass of these multiparameter families.

Two factors make the technique demonstrated here particularly suitable to software design. First, the coefficients are given as explicit symbolic expressions. Secondly, the basic stencil is used as a building block to construct finite-difference families in terms of simply expressed matrices. Thus we have developed a fairly simple algorithm to serve as the core for software to solve partial differential equations.

Finally, the ease with which stability analysis can be performed should encourage exploration of these families to discover optimal schemes.

REFERENCES

[1] W. F. Ames, *Numerical methods for partial differential equations.* New York: Barnes and Noble, 1969.

[2] L. Collatz, *The numerical treatment of differential equations.* New York: Springer-Verlag, 1960.

[3] J. Douglas, Survey of numerical methods for parabolic differential equations. In *Advances in computers,* vol. 2. New York: Academic Press (1961) 1–54.

[4] G. E. Forsythe and W. R. Wasow, *Finite difference methods for partial differential equations.* New York: Wiley, 1960.

[5] J. H. Giese and H. M. Khalil, A procedure for generating consistent families of difference operators, *Egyptian Comp. J.*, Vol. 1, No. 2 (1973) 125–136.
[6] E. Isaacson and H. B. Keller, *Analysis of numerical methods.* New York: Wiley, 1967.
[7] H. M. Khalil, *A two-parameter family of approximations to the two-dimensional heat operator.* Ph.D. Diss., Univ. of Delaware, 1969.
[8] H. M. Khalil and J. H. Giese, A two-parameter family of two-level higher order difference methods for the two-dimensional heat equation, *J. Inst. Maths. Applications* 16 (1975) 193–205.
[9] A. R. Mitchell, *Computational methods in partial differential equations.* New York: Wiley, 1969.
[10] J. Moses, MACSYMA — The fifth year. *Proc. of Eurosam.* Sweden: Royal Inst. of Tech. (1974) 105–110.
[11] R. D. Richtmyer and K. W. Morton, *Difference methods for initial-value problems.* New York: Interscience Press, 1967.
[12] V. K. Saul'yev, *Integration of equations of parabolic type by the method of nets.* New York: MacMillan, 1964.
[13] R. S. Varga, *Matrix iterative analysis.* Englewood Cliffs, N.J.: Prentice-Hall, 1962.